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THE STRUCTURE OF ELECTRO-HYDRODYNAMIC DISCONTINUITIES PRODUCED BY CHARGED GRIDS *

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The two-dimensional steady flow of a gas with a unipolar charge is discussed. The parameter of electro-hydrodynamic (EHD) interaction is assumed to be small so that the electric field does not affect the gas flow. A charged grid consisting of rectilinear wires at right angles to the plane of the flow is placed in the flow, the grid being the surface of discontinuity of the EHD parameters.

The structure of such a discontinuity in the cases where the wires are charged positively or negatively, and the charges of adjacent wires have different signs is analysed. In each such case, the grid may be partially or totally penetrable for the ions, or not penetrable at all. The charge-current characteristics of the grid are found, and the magnitude of the charge for which the grid becomes impenetrable is established. The relations which close the system of relations at a discontinuity are obtained.

1. Consider the one-dimensional flow of a two-component medium which is a mixture of gas and positive ions in the EHD-approximation, with a small interaction parameter. We direct the coordinate axis y along the flow velocity and assume that the electric field intensity has a single component parallel to the y-axis. Let the velocity of the medium v° be constant along the stream, and all other flow parameters depend on y only. In the cross-section $y=0$ we place a charged metal grid, perpendicular to the flow, which consists of parallel wires, and is sufficiently coarse so that, in practice, it does not affect the motion of the gas.

It will be shown below that depending on the magnitude and sign of the charge, the grid can be penetrable for ions partially or totally, or not penetrable at all.

We shall denote by minus or plus the values of parameters directly in front of or behind the discontinuity. Let $L, L_{E\mp}, L_{q\mp}$ be a typical geometric scale of the problem (for example, the distance of the grid from other bodies), and the typical lengths of the change in the strength of the field E, and the change in the volume density of the ion charges, q, in the flow regions in front of and behind the grid respectively. The parallel wires which form the grid are circular cylinders of radius δ, with a distance l between their axes satisfying the conditions $\min(L, L_{E\mp}, L_{q\mp}) \gg l \gg \delta$. Then, generally speaking, the grid can be regarded as a surface of discontinuity of the flow's electric parameters.

With these assumptions, the equations which describe the behaviour of electrodynamic quantities in the regions in front of and behind a discontinuity have the form

$$j = q(v^\circ + bE) = \text{const}, \quad dE/dy = 4\pi q \tag{1.1}$$

(see /1, 2/) where j is the density of the stream of ions, and b is the ion mobility, which should be constant. In addition, it is assumed that $L_{q\mp}(v^\circ + bE) \gg D$, where D is the diffusion coefficient of the ions. The electrical parameters at the discontinuity are connected by the following relations (see /2/):

$$E^+ - E^- = 4\pi\sigma, \quad j^+ - j^- = -J, \quad j^\pm = q^\pm(v^\circ + bE^\pm) \tag{1.2}$$

Here σ is the surface charge density at the discontinuity, and J is the electric current density flowing to the grid as a result of the ion deposition.

2. Let us look into the structure of the discontinuity described in the previous section. The values of $E^\mp$ introduced represent the values of the field strength at infinity in front of and behind the grid. We direct the x-axis of the Cartesian system of coordinates x and y perpendicular to the axes of the grid wires. Then all parameters of the EHD flow depend only on the coordinates x and y, and the flow of the medium is parallel to the (x, y) plane. The centre lines of the grid wires intersect the xy plane at points with coordinates $(\pm kl, 0)$, $k = 0, 1, 2, \ldots$

Let us confine ourselves to the cases most frequently realized in practice, when the

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inequalities

$$\frac{4\pi q l}{\min E^{\mp}} \ll 1, \quad \frac{D}{l \min (v^\circ + bE^{\mp})} \ll 1 \tag{2.1}$$

hold.

Then in the problem regarding the structure of the discontinuities of electric parameters caused by the grid we can ignore the volume charge of the ions and their diffusion. As a result, the relations

$$\operatorname{div} \mathbf{E} = 0, \quad \operatorname{rot} \mathbf{E} = 0, \quad \operatorname{div} \mathbf{j} = 0, \quad \mathbf{j} = q(\mathbf{v} + b\mathbf{E}) \tag{2.2}$$

will hold; here $\mathbf{v} + b\mathbf{E}$ is the velocity of the ion motion.

The field strength at infinity in front of and behind the grid can be written in the form

$$E_{\mp} = E^0 \mp 2\pi Q/l, \quad E^\circ = (E^- + E^+)/2 \tag{2.3}$$

where Q is the charge per unit length of one wire and $\sigma = Q/l$ is the mean charge per unit of grid area. It is assumed here and below in Sections 3 and 4 that Q is the same for all wires.

The case where the field produced by a coarse grid is such that it shows a marked influence on the ion motion, i.e.

$$| bE^{\mp} | \sim 2\pi b \, | Q \,|/l \sim \max (bE^\circ, v^\circ) \tag{2.4}$$

is of practical interest.

In the vicinity of each grid wire, the estimate $E_\delta \sim 2Q/\delta$ is valid for the field intensity E_δ created by the electric charges on the wires. Since the grid is assumed to be fairly coarse, and this means that $\delta \ll l$, it follows from the estimate of E_δ above and from formula (2.4) that $E_\delta \gg E^\circ$. Similarly, $E_\delta \gg E_\delta'$, where E_δ' is the field strength around any one of the wires, caused by the other wires. Therefore, in the vicinity of each wire, we can ignore the distortion of the outside field due to the electrostatic induction, and the field produced by all the other wires. Then the field outside the grid is identical with the field produced by a system of parallel infinitely thin threads with a linear charge density Q, situated on the axes of the wires and being in an external field of intensity E^0. As a result, the solution of the first two equations of (2.2) has the form (see /3/)

$$E^* \equiv E_x - iE_y = -iE^\circ + \frac{\pi}{l}\left[\frac{2Q}{z} + \sum_{k=1}^{\infty}\left(\frac{2Q}{z - k\pi} + \frac{2Q}{z + k\pi}\right)\right] = -iE^\circ + \frac{2\pi Q}{l} \operatorname{ctg} z, \quad z = \frac{\pi}{l}(x + iy) \tag{2.5}$$

(in the complex representation an asterisk denotes complex conjugation).

It follows from formula (2.4) that

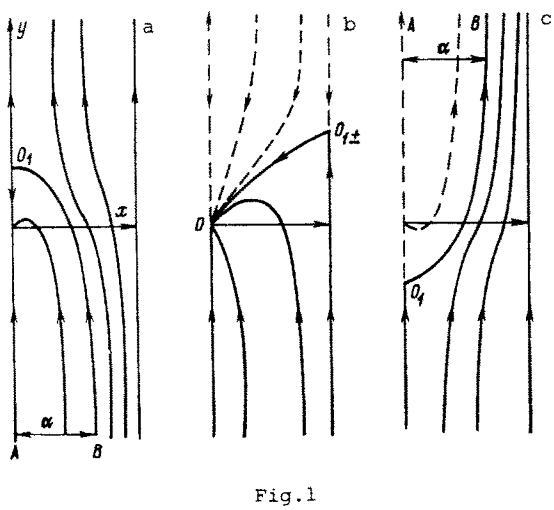

Fig.1

$$| v | \lesssim 2\pi b \, | Q \,|/l \ll 2\pi b \, | Q \,|/\delta \sim \pi b E_\delta$$

Thus, in regions close to the grid wires where a pronounced perturbation of the velocity of the hydrodynamic flow takes place, the effect of perturbation on the ion motion can generally be ignored, compared with the influence of the electric field. Consequently, henceforth, in the study of ion motion we shall assume that $v \equiv v_x + iv_y = iv^\circ = \text{const}$.

Let us introduce the complex velocity of the ions, V, and its complex potential W,

$$V^* = V_x - iV_y = -i(v^\circ + bE^\circ) + \varkappa \operatorname{ctg} z = \frac{dW}{dz}, \quad \varkappa = \frac{2\pi bQ}{l} \tag{2.6}$$

On integrating the above we obtain

$$W = \varkappa \operatorname{Ln}[C \sin z \exp(-i \operatorname{Re}_E z)], \quad \operatorname{Re}_E = (v^\circ + bE^\circ)/\varkappa \tag{2.7}$$

where C is an arbitrary constant, and Re_E is the electric Reynolds number of the grid.

From relation (2.2) and the definition of the complex potential W it follows that along the ion stream lines the density of the ion charge q and the imaginary part of the complex potential $\operatorname{Im} W$, which is a function of the ion stream, remain constant.

3. Consider the flow of a medium in the case where the grid is charged negatively ($Q < 0$). The ion streamlines inthe band $0 \leqslant x \leqslant l/2, -\infty < y < \infty$ are shown qualitatively in Fig. 1a, b. The streamlines in the band $-l/2 \leqslant x \leqslant 0, -\infty < y < \infty$ are obtained by mirror reflection with respect to the y-axis. The flow in the regions which are obtained by a shift $\pm kl$ in the direction of the x-axis is similar to the flow in the band $-l/2 \leqslant x \leqslant l/2, \ -\infty < y < \infty$, since the grid is periodic. All ions whose streamlines in Fig. 1a are in region AO_1B, reach the surfaces of wires and settle on them. We shall describe the region AO_1B as a region of ion capture by the grid in the band $0 \leqslant x \leqslant l/2, -\infty < y < \infty$. The ions whose streamlines lie to the right of the capture region AO_1B pass through the grid and go to infinity. Thus, in the case shown in Fig. 1a, the grid is partially penetrable for ions. At the same time, $q^+ = q^-$, $j^+ = q^+(v^\circ + bE^+) = q^+(v^\circ + bE^\circ + \varkappa) < j^-$.

Let us obtain an equation for the streamline which passes through the critical point O_1. The desired streamline consists of the half-line $x = 0, y > 0$, and the curve O_1B. We equate to zero the imaginary part of the complex potential of the ion velocity (2.7); then

$$C \sin z \exp(-i \operatorname{Re}_E z) = C^* \sin z^* \exp(i \operatorname{Re}_E z^*) \tag{3.1}$$

Using the condition that $z = i\pi y/l$ on that part of the streamlines which pass through the point O_1, we obtain $C^* = -C$. On solving Eq. (3.1) for y, and taking into account the definition $z = \pi(x + iy)/l$ we find the equation of the curve O_1B:

$$y = \frac{l}{2\pi} \ln \frac{\sin[(\operatorname{Re}_E - 1)\pi x/l]}{\sin[(\operatorname{Re}_E + 1)\pi x/l]} \tag{3.2}$$

The coordinates ofthe critical point O_1 at which the ion velocity is zero can be obtained from the above equation. They are

$$x_1 = 0, \quad y_1 = \frac{l}{2\pi} \ln \frac{v^\circ + bE^\circ - \varkappa}{v^\circ + bE^\circ + \varkappa} \tag{3.3}$$

The width of the region of ion capture at infinity, α_∞, can be found either from Eq. (3.2) or from the condition that all ions from the region of capture settle on the wire. Its width is

$$\alpha_\infty = \frac{2\pi b|Q|}{v^\circ + bE^\circ + |\varkappa|} = \frac{l}{1 + |\operatorname{Re}_E|} \tag{3.4}$$

When the absolute value of the charge on the wire increases, the critical point O_1 at which the ion velocity is zero, is shifted upwards, and for $Q = -l(v^\circ + bE^\circ)/(2\pi b)$ it goes to infinity. From formula (3.4) we can obtain that in this case the region of ion capture is the whole band under consideration ($\alpha_\infty = l/2$). When the absolute value of the charge is increased further, the possible value of the ion velocity at infinity behind the grid becomes negative. When this value passes through zero at infinity, on the straight lines $x = \pm l/2$ two special points $O_{1\pm}$ (instead of one O_1) are formed, which are then shifted downwards. Using the condition that the ion velocities at the points $O_{1\pm}$ are zero, we find their coordinates

$$x_{1\pm} = \pm \frac{l}{2}, \quad y_{1\pm} = \frac{l}{2\pi} \ln \frac{\varkappa - v^\circ - bE^\circ}{\varkappa + v^\circ + bE^\circ} \tag{3.5}$$

The qualitative picture of the ion streamlines of such a flow is shown in Fig. 1b. Above the line OO_{1+} there are no ions, if there are none at infinity behind the grid. When the absolute value of the charge of the wire $Q < 0$ increases, point O_{1+} is shifted downwards and as $Q \to -\infty$ it conicides with the point $(l, 0)$. In the case shown in Fig. 1b, the grid is impenetrable for the ions, with $q^+ = 0, j^+ = 0$.

4. Let us examine the flow of the medium when the grid is charged positively ($Q > 0$). Then the ion streamlines in the band $0 \leqslant x \leqslant l/2, -\infty < y < \infty$ are as illustrated in Fig. 1c. The ions which come from infinity from below intersect the grid, pass round the shadow zone AO_1B in which there are no ions, and go upwards to infinity. In this case the grid is fully penetrable for the ions, and their density on the grid is zero. It can be shown that the equation of the curve O_1B has the form (3.2) as before, where now $Q > 0$. Correspondingly, the coordinates of the critical point O_1 are described by formula (3.3). For the values $y \gg l$, the eqution of the curve O_1B (3.2), is transformed approximately to the form

$$x \simeq \alpha_\infty \left[1 - \frac{1}{\pi} \sin\left(\pi \frac{\mathrm{Re}_E - 1}{\mathrm{Re}_E + 1}\right) \exp\left(-\frac{2\pi y}{l}\right)\right] \tag{4.1}$$

where $x = \alpha_\infty$ is the straight line which limits the shadow zone at infinity; the quantity α_∞ is determined by (3.4) as before. It can be seen from formula (4.1) that $x \simeq \alpha_\infty$ when $y \gg l$. The width of the shadow zone at infinity in the band discussed is α_∞. We note that as the charge on the grid becomes greater, the width of the shadow zone increases, and for $Q = l(v^\circ + bE^\circ)/(2\pi b)$ it equals $l/2$. Here the grid totally repulses the ions, and becomes impenetrable, and as the condition at the discontinuity we must adopt $q^- = 0$.

The value of α_∞ for the shadow zone width at infinity was obtained under the assumption that the ion motion is directed by the outside electric field and is influenced by the friction force, and there is no diffusion of ions. In reality, the shadow zone at infinity vanished because of the diffusion of ions and the influence of the field created by them. We shall consider the case where the latter can be ignored. Clearly, this case is always realized for sufficiently small q. Below we give a quantitative formulation of this condition.

Let the length of the shadow zone be l_q. The ions which intersect the grid reach a distance l_q from the grid in the time $\tau \sim l_q/(v^\circ + bE^+)$. For the shadow zone to disappear at a distance l_q from the grid, because of the diffusion the ions should be displaced to a distance of the order of

$$l \sim (D\tau)^{1/2} \sim \left(\frac{Dl_q}{v^\circ + bE^+}\right)^{1/2} \tag{4.2}$$

Since, in accordance with inequality (2.1) we have $\mathrm{Pe} \equiv l(v^\circ + bE^+)/D \gg 1$, from (4.2) we obtain

$$l_q \sim l^2 (v^\circ + bE^+)/D = \mathrm{Pe}\, l \gg l \tag{4.3}$$

Thus, the process of equalizing the velocity and density of the ion charge occurs in two stages. First, at a distance $\sim l$ the ion velocity becomes practically equal to $v^\circ + bE^+$. Here the shadow zone width differs little from its limit value α_∞, and the charge density inside the zone is close to zero, and outside it equals q^-. Then at a distance of the order of $l_q \gg l$, because of diffusion, an equalization of the charge density of the ions across the band under consideration occurs, and the shadow zone disappears. Taking into account that $\mathrm{Pe} \gg 1$, this diffusion process is approximately described, under the condition that $4\pi q l_q \ll E^+)$, by the relations

$$\begin{aligned}
&(v^\circ + bE^+)\frac{\partial q}{\partial y} = D\frac{\partial^2 q}{\partial x^2} \\
&q = q^-, \quad \alpha_\infty < x \leqslant l/2, \quad y = l; \\
&q = 0, \quad 0 \leqslant x < \alpha_\infty, \quad y = l; \\
&\partial q/\partial x = 0, \quad x = 0,\ l/2, \quad y > l
\end{aligned} \tag{4.4}$$

The last two relations in (4.4) follow from the symmetry and periodicity of the grid.

The solution of problem (4.4) has the form

$$q = q^-\left(1 - \frac{2\alpha_\infty}{l}\right) - \frac{2q^-}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\exp\left[-\frac{4D\pi^2k^2(y-l)}{(v^\circ + bE^+)\,l^2}\right] \times \sin\frac{2\pi k\alpha_\infty}{l}\cos\frac{2\pi kx}{l} \tag{4.5}$$

(see /4/).

Hence it follows that for $y \gtrsim l_q$ we have $q \simeq q^-(1 - 2\alpha_\infty/l) \equiv q^+$, that is the ion charge becomes practically constant at a distance l_q from the grid. At the same time the density of the ion stream j^- also ceases to depend on x, and the relation $j^+ = j^- = \mathrm{const}$ is satisfied.

Let us estimate the length of the shadow zone l_q' when it disappears under the influence of the electric field created by the ions. Suppose that E' is the value of the field strength projection on a plane perpendicular to the stream, i.e. on the grid plane. Obviously,

$$E' \lesssim 4\pi q l, \quad l_q' \sim (v^\circ + bE^+)\frac{l}{bE'} \gtrsim \frac{v^\circ + bE^-}{4\pi b q}$$

Hence, and from expression (4.3) for l_q there follows the condition $D/(4\pi bq) \gg l^2$; if it is satisfied we have $l_q' \gg l_q$, and, therefore, the process of disappearance of the shadow zone is determined by ion diffusion.

In the cases shown in Fig. 1, the charge-current characteristics of the grid have the form

$$J \equiv j^- - j^+ = \frac{2\alpha_\infty}{l}\, j^- = \begin{cases} 0, & \varkappa \geqslant 0 \\ -2\varkappa q^-, & -(v^\circ + bE^\circ) < \varkappa < 0 \\ q^-(v^\circ + bE^\circ - \varkappa), & \varkappa \leqslant -(v^\circ + bE^\circ) \end{cases}$$

5. Consider the flow of a medium when neighbouring wires of the grid have charges of the same absolute value $Q > 0$, but of opposite sign. Let the wires whose axes intersect the

xy plane at points with coordinates $(\pm 2kl, 0)$, $k = 0, 1, 2, \ldots$, be charged positively, and those whose axes intersect the xy plane at points with coordinates $(\pm(2k+1)\,l, 0)$ be charged negatively. We shall consider the flow of the medium in the band $0 \leqslant x \leqslant l$, $-\infty < y < \infty$. The flow in the band $-l \leqslant x \leqslant 0, -\infty < y < \infty$ is obtained by mirror reflection in the y-axis. The flow in the regions obtained by a shift of $\pm 2kl$ in the direction of the x-axis is similar to the flow in the band $-l \leqslant x \leqslant l$, $-\infty < y < \infty$ because the grid is periodic.

In the approximation described in Section 2, the electric field strength in the flow is determined by the formula

$$E^* \equiv E_x - iE_y = -iE^c + \frac{\pi}{l}\left[\frac{2Q}{z} + \sum_{k=1}^{\infty}(-1)^k\left(\frac{2Q}{z-k\pi} + \frac{2Q}{z+k\pi}\right)\right] = -iE^c + \frac{\pi}{l}\,\frac{2Q}{\sin z}$$

(see /3/.

The complex ion velocity V and its complex potential W are

$$V^* = V_x - iV_y = -i\,(v^\circ + bE^\circ) + \frac{\varkappa}{\sin z}$$

$$W = \varkappa\,\mathrm{Ln}\left[C\,\mathrm{tg}\,\frac{z}{2}\exp(-i\,\mathrm{Re}_E z)\right], \quad C = \mathrm{const}$$

The qualitative picture of the ion streamlines in the band $0 \leqslant x \leqslant l$, $-\infty < y < \infty$ is shown in Fig. 2a. The ions whose streamlines lie to the

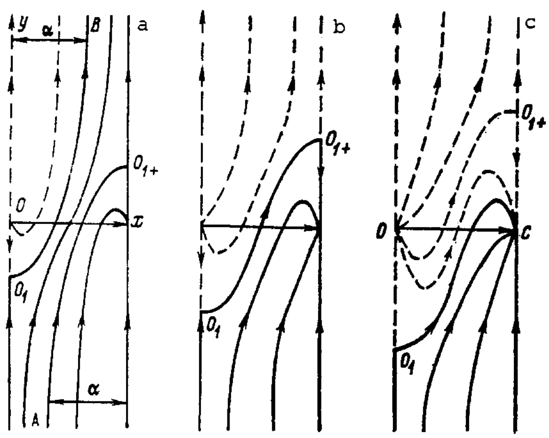

Fig. 2

left of the line AO_{1+} pass through the grid, bending round the shadow zone OO_1B where there are no ions, and go to infinity. All ions whose streamlines lie to the right of the line AO_{1+} reach the wires and settle on them. In the case shown in Fig. 1a, the grid with bipolar charges is partially penetrable by the ions.

By the operation used in Sections 2 and 3 we obtain the equations of streamlines which pass through the critical points O_1 and $O_{1\pm}$ lying on the straight lines $x = 0$ and $x = \pm l$, and the coordinates of these points,

$$x = al, \quad (-1)^{a+2} y < 0: \quad y = \frac{l}{\pi}\ln\left[(1 + \chi_a{}^2)^{1/2} - (-1)^{a+2}\chi_a\right]$$

$$\chi_a \equiv \sin\pi\left(\frac{x}{l} - a\right)\frac{\cos[\mathrm{Re}_E\pi(x/l - a)]}{\sin[\mathrm{Re}_E\pi(x/l - a)]}$$

$$x_a = al, \quad y_a = \frac{l}{\pi}\ln\left[\left(\left(\frac{\varkappa}{v^c + bE^c}\right)^2 + 1\right)^{1/2} - (-1)^{a+2}\frac{\varkappa}{v^c + bE^c}\right]$$

Here $a = 0, \pm 1$ for the points O_1 and $O_{1\pm}$ respectively.

The width of the region of ion capture in the band discussed, from which the ions settle on the negatively charged wire, equals the width of the shadow zone which is formed behind the positively charged wire, and is found from the formula

$$\alpha_\infty = \frac{2\pi bQ}{v^c + bE^c} \equiv \frac{l}{\mathrm{Re}_E} \tag{5.1}$$

This formula, as was the corresponding formula in Section 4, is obtained under the assumption that there is no ion diffusion. In reality, the shadow zone at infinity will disappear owing to the ion diffusion. By the argument used in Section 4, we conclude that the process of equalizing the ion charge behind the grid is described by formula (4.5), in which we can replace E^+ and l by E^c and $2l$ respectively. At the same time, in this case the shadow zone width α_∞ which occurs in Eq.(4.5) is determined by formula (5.1). At a distance l_q from the grid the density of the ion charge q becomes practically constant and equals $q^+ \simeq q^- (1 - \alpha_\infty/l)$.

The density of the ion stream j^+ becomes constant, but unlike the case described in Section 4, some of the ions settle on the grid and, therefore, the relation $j^+ = q^+ (v^\circ + bE^\circ) < j^-$ is satisfied.

When the absolute value of the charge on the wires increases the critical point O_1 shifts downwards, and the critical points $O_{1\pm}$ shift upwards. The width of the capture region and that of the shadow zone increase, and for $Q = l(v^0 + bE^0)/(2\pi b)$ become equal to l. At the same time the curves O_1B and AO_{1+} merge, and one streamline, given by the equation

$$x = 0,\quad y < 0;\quad x = \pm l,\quad y > 0$$
$$y = \frac{l}{\pi}\left[\left(\cos^2\frac{\pi x}{l} + 1\right)^{1/2} - \cos\frac{\pi x}{l}\right]$$

passes the critical points $O_1, O_{1\pm}$

The qualitative picture of the ion streamlines of such a flow is shown in Fig. 2b. The grid becomes impenetrable for the ions and all of them settle on the negatively charged wires. Above the line O_1O_{1+} there are no ions, and therefore $q^- = j^- = 0$.

Figure 2c shows the streamlines of the medium's flow when the absolute value of the charge on the grid wires is $Q > l(v^0 + bE^0)/(2\pi b)$. It can be seen in Fig. 2c. that one line which in Fig. 2a passes through the critical points O_1, O_{1+} splits again into two lines: O_1C and OO_{1+}. There are no ions above the line O_1C. Consequently, in this case also the grid is impenetrable for the ions, and $q^- = j^- = 0$.

In the cases shown in Fig.2, the charge-current characteristics have the form

$$J \equiv j^- - j^+ = j^- \frac{\alpha_\infty^{\pm}}{l} = \begin{cases} \varkappa q^-, & 0 < \varkappa < v^\circ + bE^\circ \\ q^- (v^\circ + bE^\circ), & \varkappa \geqslant (v^\circ + bE^\circ) \end{cases}$$

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